International Journal of Theoretical Physics, Vol. 40, No. 8, 2001

Finslerian *N*-Spinors: Algebra

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Received May 5, 2000

New mathematical objects called Finslerian N-spinors are discussed. The Finslerian N-spinor algebra is developed. It is found that Finslerian N-spinors are associated with an N^2-dimensional flat Finslerian space. A generalization of the epimorphism SL $(2, \mathbb{C}) \to O_+^{\uparrow}(1, 3)$ to a case of the group SL$(N, \mathbb{C})$ is constructed. Particular examples of Finslerian N-spinors for $N = 2, 3$ are considered in detail.

1. INTRODUCTION

Spinors as geometrical objects were discovered by É. Cartan in 1913 (Cartan, 1913). One decade later, Pauli (1927) and Dirac (1928) rediscovered spinors in connection with the problem of describing the spin of an electron. From that time, spinors are intensively used in mathematics and physics.

In the classical works (Brauer and Weyl, 1935; Cartan, 1938), a concept of the Cartan's 2-spinor was generalized and the theory of spinors in an arbitrary n-dimensional pseudo-Euclidean space was constructed. In this paper, another generalization of 2-spinors is proposed which leads to the Finslerian geometry. Originally, such a generalization appeared within the so-called *relational theory of space-time* (Solov'yov, 1996; Vladimirov, 1996). However, the corresponding mathematical scheme also has an independent meaning and will be presented later.

In the next section, we shall develop a general algebraic formalism of Finslerian N-spinors. The subsequent sections deal with the theory of the simplest Finslerian 2- and 3-spinors. Conclusion contains some remarks concerning the obtained results.

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2. GENERAL FORMALISM

Let $\mathbb{FS}^N$ be a vector space of $N > 1$ dimensions over $\mathbb{C}$ and

$$[\cdot, \cdot, \ldots, \cdot] : \underbrace{\mathbb{FS}^N \times \mathbb{FS}^N \times \cdots \times \mathbb{FS}^N}_{N \text{ multiplicands}} \to \mathbb{C} \tag{1}$$

be a nonzero antisymmetric N-linear functional on $\mathbb{FS}^N$. The latter means

(i) there exist $\boldsymbol{\xi}_0, \boldsymbol{\eta}_0, \ldots, \boldsymbol{\lambda}_0 \in \mathbb{FS}^N$ such that

$$[\boldsymbol{\xi}_0, \boldsymbol{\eta}_0, \ldots, \boldsymbol{\lambda}_0] = z_0 \neq 0; \tag{2}$$

(ii) for any $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_N \in \mathbb{FS}^N$,

$$[\boldsymbol{\xi}_a, \boldsymbol{\xi}_b, \ldots, \boldsymbol{\xi}_c] = \varepsilon_{ab\cdots c}[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_N],$$

where $a, b, \ldots, c = 1, 2, \ldots, N$ and $\varepsilon_{ab\cdots c}$ is the N-dimensional Levi-Civita symbol with the ordinary normalization $\varepsilon_{12\cdots N} = 1$;

(iii) for any $\boldsymbol{\xi}_1, \boldsymbol{\eta}_1, \boldsymbol{\xi}_2, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\xi}_N, \boldsymbol{\eta}_N \in \mathbb{FS}^N$ and $z \in \mathbb{C}$,

$$[\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_a + \boldsymbol{\eta}_a, \ldots, \boldsymbol{\xi}_N] = [\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_a, \ldots, \boldsymbol{\xi}_N] + [\boldsymbol{\xi}_1, \ldots, \boldsymbol{\eta}_a, \ldots, \boldsymbol{\xi}_N],$$

$$[\boldsymbol{\xi}_1, \ldots, z\boldsymbol{\xi}_a, \ldots, \boldsymbol{\xi}_N] = z[\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_a, \ldots, \boldsymbol{\xi}_N],$$

where a takes the values $1, 2, \ldots, N$.

We shall use the following terminology: The space $\mathbb{FS}^N$ equipped with the functional (1) having the properties (i), (ii), and (iii) is called the *space of Finslerian N-spinors*. The complex number $[\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots, \boldsymbol{\lambda}]$ is respectively called the *scalar N-product* of the Finslerian N-spinors $\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots, \boldsymbol{\lambda} \in \mathbb{FS}^N$.

It should be noted that $\boldsymbol{\xi}_0, \boldsymbol{\eta}_0, \ldots, \boldsymbol{\lambda}_0$ are linearly independent. Indeed, if those were linearly dependent, one of the Finslerian N-spinors $\boldsymbol{\xi}_0, \boldsymbol{\eta}_0, \ldots, \boldsymbol{\lambda}_0$ would be a linear combination of the others and, in accordance with (ii) and (iii), the scalar N-product $[\boldsymbol{\xi}_0, \boldsymbol{\eta}_0, \ldots, \boldsymbol{\lambda}_0]$ would be equal to zero. However, this is in contradiction with (2). Thus, $\boldsymbol{\xi}_0, \boldsymbol{\eta}_0, \ldots, \boldsymbol{\lambda}_0$ are linearly independent, i.e., form a basis in $\mathbb{FS}^N$.

Let us introduce the notation $\boldsymbol{\epsilon}_1 = \boldsymbol{\xi}_0, \boldsymbol{\epsilon}_2 = \boldsymbol{\eta}_0, \ldots, \boldsymbol{\epsilon}_N = \boldsymbol{\lambda}_0/z_0$. It is evident that the set $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \ldots, \boldsymbol{\epsilon}_N\}$ is a basis in $\mathbb{FS}^N$. Because of (2) and (iii), its elements satisfy the condition

$$[\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \ldots, \boldsymbol{\epsilon}_N] = 1. \tag{3}$$

We shall call such a basis *canonical*.

Let $\boldsymbol{\epsilon}'_1, \boldsymbol{\epsilon}'_2, \ldots, \boldsymbol{\epsilon}'_N$ be arbitrary Finslerian N-spinors and

$$\boldsymbol{\epsilon}'_a = c_a^b \boldsymbol{\epsilon}_b \tag{4}$$

be their expansions into the canonical basis $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \ldots, \boldsymbol{\epsilon}_N\}$; here $a, b = 1, 2, \ldots, N$, $c_a^b \in \mathbb{C}$, and the summation is taken over the repeating index b. With the help of (ii), (iii), (3), and (4), we find

$$[\boldsymbol{\epsilon}'_1, \boldsymbol{\epsilon}'_2, \ldots, \boldsymbol{\epsilon}'_N] = \det \mathrm{C}_N, \tag{5}$$

where $\mathrm{C}_N = \|c_a^b\|$. Since linear (in)dependence of $\boldsymbol{\epsilon}'_1, \boldsymbol{\epsilon}'_2, \ldots, \boldsymbol{\epsilon}'_N$ is equivalent to that of columns of the complex $N \times N$ matrix C_N, the set $\{\boldsymbol{\epsilon}'_1, \boldsymbol{\epsilon}'_2, \ldots, \boldsymbol{\epsilon}'_N\}$ is a basis in $\mathbb{FS}^N$ if and only if $\det \mathrm{C}_N \neq 0$. Moreover, it follows from (5) that $\{\boldsymbol{\epsilon}'_1, \boldsymbol{\epsilon}'_2, \ldots, \boldsymbol{\epsilon}'_N\}$ is a canonical one when $\det \mathrm{C}_N = 1$. Thus, if C_N runs the group $\mathrm{SL}(N, \mathbb{C})$ of unimodular complex $N \times N$ matrices, then $\{\boldsymbol{\epsilon}'_1, \boldsymbol{\epsilon}'_2, \ldots, \boldsymbol{\epsilon}'_N\}$ runs the set $\mathbf{E}(\mathbb{FS}^N)$ of canonical bases in $\mathbb{FS}^N$.

Let us express the scalar N-product of Finslerian N-spinors in terms of their components with respect to *any* canonical basis $\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_N\} \in \mathbf{E}(\mathbb{FS}^N)$. By using (ii), (iii), (3), and the expansions $\boldsymbol{\xi} = \xi^a \boldsymbol{\epsilon}_a$, $\boldsymbol{\eta} = \eta^b \boldsymbol{\epsilon}_b, \ldots, \boldsymbol{\lambda} = \lambda^c \boldsymbol{\epsilon}_c$, it is possible to show that

$$[\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots, \boldsymbol{\lambda}] = \varepsilon_{ab\cdots c} \xi^a \eta^b \cdots \lambda^c, \tag{6}$$

where $\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots, \boldsymbol{\lambda} \in \mathbb{FS}^N$, $\xi^a, \eta^b, \ldots, \lambda^c \in \mathbb{C}$, $a, b, \ldots, c = 1, 2, \ldots, N$. In (6) as well as in the following formulas of this paper, the summation is taken over all the repeating indices. It is clear that the scalar N-product (6) is zero if and only if $\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots, \boldsymbol{\lambda}$ are linearly dependent Finslerian N-spinors.

Let us consider a mapping

$$\begin{aligned} &\boldsymbol{S} : \mathbf{E}(\mathbb{FS}^N) \to \mathbb{C}^{N^{k+l+m+n}}, \\ &\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_N\} \mapsto \boldsymbol{S}\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_N\} = \left(S^{b_1\cdots b_k \dot{c}_1\cdots\dot{c}_l}_{a_1\cdots a_m \dot{d}_1\cdots\dot{d}_n}\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_N\}\right) \end{aligned} \tag{7}$$

such that

$$\begin{aligned} S^{b_1\cdots b_k \dot{c}_1\cdots\dot{c}_l}_{a_1\cdots a_m \dot{d}_1\cdots\dot{d}_n}\{\boldsymbol{\epsilon}'_1, \ldots, \boldsymbol{\epsilon}'_N\} = {} & c_{a_1}^{e_1} \cdots c_{a_m}^{e_m} \overline{c_{\dot{d}_1}^{\dot{h}_1}} \cdots \overline{c_{\dot{d}_n}^{\dot{h}_n}} d_{f_1}^{b_1} \cdots d_{f_k}^{b_k} \overline{d_{\dot{g}_1}^{\dot{c}_1}} \cdots \overline{d_{\dot{g}_l}^{\dot{c}_l}} \\ & \times S^{f_1\cdots f_k \dot{g}_1\cdots\dot{g}_l}_{e_1\cdots e_m \dot{h}_1\cdots\dot{h}_n}\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_N\} \end{aligned} \tag{8}$$

for any two canonical bases $\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_N\}, \{\boldsymbol{\epsilon}'_1, \ldots, \boldsymbol{\epsilon}'_N\} \in \mathbf{E}(\mathbb{FS}^N)$ whose elements are connected by the relations (4). Here all the indices (both ordinary and dotted) run independently from 1 to N, the overlines denote complex conjugating, d_b^a are the complex numbers satisfying the conditions $c_a^b d_c^a = \delta_c^b$ (δ_c^b is the Kronecker symbol), $\det \|c_b^a\| = \det \|d_b^a\| = 1$, and $k, l, m,$ and n are nonnegative integers.

Every mapping (7), which possesses the property (8), is called a *Finslerian N-spintensor of a valency* $\left[\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right]$. The addition and multiplication of such N-spintensors are defined in the standard way: if $\boldsymbol{S}$ and $\boldsymbol{T}$ have the valency $\left[\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right]$

while $\boldsymbol{U}$ has the valency $[{}^{p}_{r}\ {}^{q}_{s}]$, then

$$(S+T)^{b_1\cdots b_k\dot{c}_1\cdots\dot{c}_l}_{a_1\cdots a_m\dot{d}_1\cdots\dot{d}_n}\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\} = S^{b_1\cdots b_k\dot{c}_1\cdots\dot{c}_l}_{a_1\cdots a_m\dot{d}_1\cdots\dot{d}_n}\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\} + T^{b_1\cdots b_k\dot{c}_1\cdots\dot{c}_l}_{a_1\cdots a_m\dot{d}_1\cdots\dot{d}_n}\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\}$$

are the components of the sum $\boldsymbol{S}+\boldsymbol{T}$ while

$$(S\otimes U)^{b_1\cdots b_{k+p}\dot{c}_1\cdots\dot{c}_{l+q}}_{a_1\cdots a_{m+r}\dot{d}_1\cdots\dot{d}_{n+s}}\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\} = S^{b_1\cdots b_k\dot{c}_1\cdots\dot{c}_l}_{a_1\cdots a_m\dot{d}_1\cdots\dot{d}_n}\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\} \times U^{b_{k+1}\cdots b_{k+p}\dot{c}_{l+1}\cdots\dot{c}_{l+q}}_{a_{m+1}\cdots a_{m+r}\dot{d}_{n+1}\cdots\dot{d}_{n+s}}\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\}$$

are those of the product $\boldsymbol{S}\otimes\boldsymbol{U}$ with respect to an arbitrary canonical basis $\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\}\in\mathbf{E}(\mathbb{FS}^N)$. Notice that all Finslerian N-spintensors of the valency $[{}^{k}_{m}\ {}^{l}_{n}]$ form an $N^{k+l+m+n}$-dimensional vector space over $\mathbb{C}$.

Let Herm(N) be an N^2*-dimensional vector space over* $\mathbb{R}$ consisting of Finslerian N-spintensors $\boldsymbol{X}$ of the valency $[{}^{1}_{0}\ {}^{1}_{0}]$ whose components satisfy the Hermitian symmetry conditions

$$X^{b\dot{c}}\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\} = \overline{X^{c\dot{b}}\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\}} \tag{9}$$

for any $\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\}\in\mathbf{E}(\mathbb{FS}^N)$. Besides, let $\{\boldsymbol{E}_0,\boldsymbol{E}_1,\ldots,\boldsymbol{E}_{N^2-1}\}$ be a basis in Herm(N) and $\{\boldsymbol{\epsilon}_1,\boldsymbol{\epsilon}_2,\ldots,\boldsymbol{\epsilon}_N\}$ be a canonical one in $\mathbb{FS}^N$. With each $\{\boldsymbol{\epsilon}'_1,\boldsymbol{\epsilon}'_2,\ldots,\boldsymbol{\epsilon}'_N\}\in\mathbf{E}(\mathbb{FS}^N)$, we associate a basis $\{\boldsymbol{E}'_0,\boldsymbol{E}'_1,\ldots,\boldsymbol{E}'_{N^2-1}\}$ in Herm(N) such that

$$E'^{b\dot{c}}_{\alpha}\{\boldsymbol{\epsilon}'_1,\ldots,\boldsymbol{\epsilon}'_N\} = E^{b\dot{c}}_{\alpha}, \tag{10}$$

where $E^{b\dot{c}}_{\alpha} = E^{b\dot{c}}_{\alpha}\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\}$ and $\alpha = 0,1,\ldots,N^2-1$. In other words, (10) defines the mapping $\{\boldsymbol{\epsilon}'_1,\boldsymbol{\epsilon}'_2,\ldots,\boldsymbol{\epsilon}'_N\}\mapsto\{\boldsymbol{E}'_0,\boldsymbol{E}'_1,\ldots,\boldsymbol{E}'_{N^2-1}\}$ of $\mathbf{E}(\mathbb{FS}^N)$ into the set of all bases in Herm(N). However,

$$E'^{b\dot{c}}_{\alpha}\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\} = c^b_f\overline{c^{\dot{c}}_{\dot{g}}}E'^{f\dot{g}}_{\alpha}\{\boldsymbol{\epsilon}'_1,\ldots,\boldsymbol{\epsilon}'_N\} \tag{11}$$

(compare it with (8)). Because of (10) and (11), we obtain

$$E'^{b\dot{c}}_{\alpha}\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_N\} = c^b_f\overline{c^{\dot{c}}_{\dot{g}}}E^{f\dot{g}}_{\alpha}. \tag{12}$$

Let us consider the following expansions:

$$\boldsymbol{E}'_{\alpha} = L(\mathrm{C}_N)^{\beta}_{\alpha}\boldsymbol{E}_{\beta}, \tag{13}$$

where $L(\mathrm{C}_N)^{\beta}_{\alpha}\in\mathbb{R}$ and $\alpha,\beta = 0,1,\ldots,N^2-1$. In order to find $L(\mathrm{C}_N)^{\beta}_{\alpha}$ as the functions of c^a_b, it is useful to introduce N^2 Finslerian N-spintensors $\boldsymbol{E}^{\alpha}$ of the valency $[{}^{0}_{1}\ {}^{0}_{1}]$ such that

$$\text{contraction}\,(\boldsymbol{E}^{\alpha}\otimes\boldsymbol{E}_{\beta}) = \delta^{\alpha}_{\beta}. \tag{14}$$

It is easy to show that $\boldsymbol{E}^\alpha$ exist, are unique, and $E^\alpha_{b\dot{c}} = \overline{E^\alpha_{c\dot{b}}}$ with the notation $E^\alpha_{b\dot{c}} = E^\alpha_{b\dot{c}}\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_N\}$. Using (13) and (14), we can write

$$L(\mathrm{C}_N)^\alpha_\beta = \text{contraction}\,(\boldsymbol{E}^\alpha \otimes \boldsymbol{E}'_\beta). \tag{15}$$

On the other hand, (12) implies

$$\text{contraction}\,(\boldsymbol{E}^\alpha \otimes \boldsymbol{E}'_\beta) = E^\alpha_{b\dot{c}} c^b_f \overline{c^{\dot{c}}_{\dot{g}}} E_\beta^{f\dot{g}}. \tag{16}$$

Thus, according to (15) and (16),

$$L(\mathrm{C}_N)^\alpha_\beta = E^\alpha_{b\dot{c}} c^b_f \overline{c^{\dot{c}}_{\dot{g}}} E_\beta^{f\dot{g}}. \tag{17}$$

Let $\mathrm{E}^\alpha = \|E^\alpha_{\dot{c}b}\|$, $\mathrm{E}_\beta = \|E_\beta^{f\dot{g}}\|$, and $\mathrm{E}'_\beta = \|E'^{f\dot{g}}_\beta\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_N\}\|$. Then, it is possible to rewrite (12) and (17) in the matrix form respectively as

$$\mathrm{E}'_\beta = \mathrm{C}_N \mathrm{E}_\beta \mathrm{C}_N^+ \tag{18}$$

and

$$L(\mathrm{C}_N)^\alpha_\beta = \operatorname{trace}\big(\mathrm{E}^\alpha \mathrm{C}_N \mathrm{E}_\beta \mathrm{C}_N^+\big), \tag{19}$$

where the cross denotes Hermitian conjugating. However, it follows from (13) that $\mathrm{E}'_\beta = L(\mathrm{C}_N)^\gamma_\beta \mathrm{E}_\gamma$. Therefore,

$$\mathrm{C}_N \mathrm{E}_\beta \mathrm{C}_N^+ = L(\mathrm{C}_N)^\gamma_\beta \mathrm{E}_\gamma. \tag{20}$$

Taking into account (19) and (20), we immediately obtain

$$L(\mathrm{B}_N\mathrm{C}_N)^\alpha_\beta = \operatorname{trace}\big(\mathrm{E}^\alpha \mathrm{B}_N \mathrm{C}_N \mathrm{E}_\beta \mathrm{C}_N^+ \mathrm{B}_N^+\big) = L(\mathrm{B}_N)^\alpha_\gamma L(\mathrm{C}_N)^\gamma_\beta \tag{21}$$

for any $\mathrm{B}_N, \mathrm{C}_N \in \mathrm{SL}(N, \mathbb{C})$.

Let $L(\mathrm{C}_N) = \|L(\mathrm{C}_N)^\alpha_\beta\|$ and $\mathrm{FL}(N^2, \mathbb{R}) = \{L(\mathrm{C}_N) \mid \mathrm{C}_N \in \mathrm{SL}(N, \mathbb{C})\}$. In these terms, (21) means that $\mathrm{FL}(N^2, \mathbb{R})$ is a group with respect to the matrix multiplication and the mapping

$$L\ :\ \mathrm{SL}(N, \mathbb{C}) \to \mathrm{FL}\,(N^2, \mathbb{R}), \qquad \mathrm{C}_N \mapsto L(\mathrm{C}_N) \tag{22}$$

is a group epimorphism so that, in particular, $L(1_N) = 1_{N^2}$(1_N, 1_{N^2} are the identity matrices of the corresponding orders) and $L(\mathrm{C}_N^{-1}) = L(\mathrm{C}_N)^{-1}$. It is easy to prove that the kernel of the epimorphism (22) has the form

$$\ker L = \big\{e^{i\frac{2\pi k}{N}} 1_N \mid k = 0, 1, \ldots, N-1\big\}. \tag{23}$$

Let us return to the relations (13). Since both $\{\boldsymbol{E}_0, \ldots, \boldsymbol{E}_{N^2-1}\}$ and $\{\boldsymbol{E}'_0, \ldots, \boldsymbol{E}'_{N^2-1}\}$ are bases in $\mathrm{Herm}(N)$, any vector $\boldsymbol{X} \in \mathrm{Herm}(N)$ can be expanded in the following two ways

$$\boldsymbol{X} = X^\alpha \boldsymbol{E}_\alpha = X'^\beta \boldsymbol{E}'_\beta, \tag{24}$$

where X^α, $X'^\beta \in \mathbb{R}$. It is obvious that $X'^\beta = L(\mathrm{C}_N^{-1})_\alpha^\beta X^\alpha$. On the other hand, $X^{b\dot{c}}\{\epsilon_1, \ldots, \epsilon_N\} = c_f^b \overline{c_{\dot{g}}^{\dot{c}}} X^{f\dot{g}}\{\epsilon'_1, \ldots, \epsilon'_N\}$ or, what is the same,

$$\|X^{b\dot{c}}\{\epsilon_1, \ldots, \epsilon_N\}\| = \mathrm{C}_N \|X^{f\dot{g}}\{\epsilon'_1, \ldots, \epsilon'_N\}\| \mathrm{C}_N^+. \tag{25}$$

Remembering that $\det \mathrm{C}_N = 1$ and calculating the determinant of (25), we see that

$$\det \|X^{b\dot{c}}\{\epsilon_1, \ldots, \epsilon_N\}\| = \det \|X^{f\dot{g}}\{\epsilon'_1, \ldots, \epsilon'_N\}\| \tag{26}$$

for *any* $\{\epsilon'_1, \ldots, \epsilon'_N\} \in \mathbf{E}(\mathbb{FS}^N)$. Hence, (26) gives an invariant numerical characteristic of the vector $\boldsymbol{X}$, which is naturally denoted by $\det \boldsymbol{X}$. Notice that $\det \boldsymbol{X} = \det \|X^{b\dot{c}}\{\epsilon_1, \ldots, \epsilon_N\}\| \in \mathbb{R}$ as it follows from (9).

Thus, without loss of generality, it is possible to calculate $\det \boldsymbol{X}$ with respect to the basis $\{\epsilon_1, \ldots, \epsilon_N\} \in \mathbf{E}(\mathbb{FS}^N)$. According to (24) and (26), $\det \boldsymbol{X} = \det(X^\alpha \mathrm{E}_\alpha) = \det(X'^\beta \mathrm{E}'_\beta)$. However, (18) implies $\det(X'^\beta \mathrm{E}'_\beta) = \det(\mathrm{C}_N X'^\beta \mathrm{E}_\beta \mathrm{C}_N^+) = \det(X'^\beta \mathrm{E}_\beta)$. Therefore,

$$\det \boldsymbol{X} = \det(X^\alpha \mathrm{E}_\alpha) = \det(X'^\alpha \mathrm{E}_\alpha). \tag{27}$$

At the same time,

$$\det(X^\alpha \mathrm{E}_\alpha) = G_{\alpha\beta\cdots\gamma} \underbrace{X^\alpha X^\beta \cdots X^\gamma}_{N \text{ multiplicands}}, \tag{28}$$

where the real coefficients $G_{\alpha\beta\cdots\gamma}$ are completely determined by the choice of the basis $\{\boldsymbol{E}_0, \ldots, \boldsymbol{E}_{N^2-1}\}$ in $\mathrm{Herm}(N)$. Because of (27) and (28),

$$\det \boldsymbol{X} = G_{\alpha\beta\cdots\gamma} X^\alpha X^\beta \cdots X^\gamma = G_{\alpha\beta\cdots\gamma} X'^\alpha X'^\beta \cdots X'^\gamma, \tag{29}$$

i.e., $\det \boldsymbol{X}$ is *forminvariant* under transformations of the group $\mathrm{FL}(N^2, \mathbb{R})$. Notice that (29) is valid for *any* basis $\{\boldsymbol{E}'_0, \ldots, \boldsymbol{E}'_{N^2-1}\}$ whose elements are connected with those of $\{\boldsymbol{E}_0, \ldots, \boldsymbol{E}_{N^2-1}\}$ by the relations (13).

Denoting $\det \boldsymbol{X}$ by $\boldsymbol{X}^N$ and using (28), we get (with respect to the basis $\{\boldsymbol{E}_0, \ldots, \boldsymbol{E}_{N^2-1}\}$)

$$\boldsymbol{X}^N = G_{\alpha\beta\cdots\gamma} X^\alpha X^\beta \cdots X^\gamma, \tag{30}$$

where $G_{\alpha\beta\ldots\gamma}$ are symmetric in all the indices and do not depend on the choice of any canonical basis in $\mathbb{FS}^N$. Thus, (30) correctly defines the structure of an N^2*-dimensional flat Finslerian space* on $\mathrm{Herm}(N)$ so that $\boldsymbol{X}^N$ is the Nth power of the Finslerian length of the vector $\boldsymbol{X} \in \mathrm{Herm}(N)$ (Finsler, 1918). It should be noted that, in general, the homogeneous algebraic form (30) is not positive-definite.

In the next two sections, we shall illustrate the above formalism by the simplest examples of Finslerian 2- and 3-spinors.

3. FINSLERIAN 2-SPINORS

Let us consider the case when $N = 2$. In this case, function (1) is the usual symplectic scalar multiplication on $\mathbb{FS}^2$. Therefore, $\mathbb{FS}^2$ is isomorphic to the space $\mathbb{S}^2$ of standard 2-spinors (Penrose and Rindler, 1986) so that Finslerian 2-spinors are *identical* to Weyl ones. Here, we reproduce some essential information on 2-spinors which will be necessary in the next section of this paper.

First of all, for any $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2\}, \{\boldsymbol{\epsilon}'_1, \boldsymbol{\epsilon}'_2\} \in \mathbf{E}(\mathbb{FS}^2)$ and $\boldsymbol{\xi} = \xi^a \boldsymbol{\epsilon}_a = \xi'^b \boldsymbol{\epsilon}'_b \in \mathbb{FS}^2$, (4) implies

$$\xi'^a = d^a_b \xi^b, \tag{31}$$

where $\xi'^a, \xi^b \in \mathbb{C}$, $c^a_b d^b_c = \delta^a_c$, and $a, b, c = 1, 2$. Of course, $\mathrm{C}_2, \mathrm{D}_2 \in \mathrm{SL}(2, \mathbb{C})$ and $\mathrm{D}_2 = \mathrm{C}_2^{-1}$ with the notation $\mathrm{C}_2 = \|c^a_b\|$, $\mathrm{D}_2 = \|d^a_b\|$. In the same way, (6) gives

$$[\boldsymbol{\xi}, \boldsymbol{\eta}] = \varepsilon_{ab} \xi^a \eta^b = \xi^1 \eta^2 - \xi^2 \eta^1 \tag{32}$$

for the scalar product of arbitrary 2-spinors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ with respect to a basis $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2\} \in \mathbf{E}(\mathbb{FS}^2)$.

Let us assume

$$\mathrm{E}^\alpha = \frac{1}{2}\sigma^\alpha, \qquad \mathrm{E}_\beta = \sigma_\beta, \tag{33}$$

where $\alpha, \beta = 0, 1, 2, 3$, $\sigma^\alpha = \sigma_\alpha$, and

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{34}$$

are the identity and Pauli matrices. Since trace $(\sigma^\alpha \sigma_\beta) = 2\delta^\alpha_\beta$, this choice guarantees correctness of (14). It follows from (13), (21), and (24) that

$$X'^\alpha = L(\mathrm{D}_2)^\alpha_\beta X^\beta \tag{35}$$

for any 4-vector $\boldsymbol{X} \in \mathrm{Herm}(2)$. Using (19), (33), and (34), we obtain

$$L(\mathrm{D}_2)^\alpha_\beta = \frac{1}{2}\mathrm{trace}\big(\sigma^\alpha \mathrm{D}_2 \sigma_\beta \mathrm{D}_2^+\big) \tag{36}$$

or, in the explicit form,

$$L(\mathrm{D}_2)^0_0 = \frac{1}{2}\big(d^1_1 \overline{d^{\dot{1}}_{\dot{1}}} + d^1_2 \overline{d^{\dot{1}}_{\dot{2}}} + d^2_1 \overline{d^{\dot{2}}_{\dot{1}}} + d^2_2 \overline{d^{\dot{2}}_{\dot{2}}}\big),$$

$$L(\mathrm{D}_2)^0_1 = \frac{1}{2}\big(d^1_1 \overline{d^{\dot{1}}_{\dot{2}}} + d^2_1 \overline{d^{\dot{2}}_{\dot{2}}} + d^1_2 \overline{d^{\dot{1}}_{\dot{1}}} + d^2_2 \overline{d^{\dot{2}}_{\dot{1}}}\big),$$

$$L(\mathrm{D}_2)^0_2 = \frac{i}{2}\big(d^1_2 \overline{d^{\dot{1}}_{\dot{1}}} + d^2_2 \overline{d^{\dot{2}}_{\dot{1}}} - d^1_1 \overline{d^{\dot{1}}_{\dot{2}}} - d^2_1 \overline{d^{\dot{2}}_{\dot{2}}}\big),$$

$$
\begin{aligned}
L(\mathrm{D}_2)^0_3 &= \frac{1}{2}\left(d^1_1\overline{d^{\dot{1}}_{\dot{1}}} + d^2_1\overline{d^{\dot{2}}_{\dot{1}}} - d^1_2\overline{d^{\dot{1}}_{\dot{2}}} - d^2_2\overline{d^{\dot{2}}_{\dot{2}}}\right),\\
L(\mathrm{D}_2)^1_0 &= \frac{1}{2}\left(d^1_1\overline{d^{\dot{2}}_{\dot{1}}} + d^2_1\overline{d^{\dot{1}}_{\dot{1}}} + d^1_2\overline{d^{\dot{2}}_{\dot{2}}} + d^2_2\overline{d^{\dot{1}}_{\dot{2}}}\right),\\
L(\mathrm{D}_2)^1_1 &= \frac{1}{2}\left(d^1_1\overline{d^{\dot{2}}_{\dot{2}}} + d^2_1\overline{d^{\dot{1}}_{\dot{2}}} + d^1_2\overline{d^{\dot{2}}_{\dot{1}}} + d^2_2\overline{d^{\dot{1}}_{\dot{1}}}\right),\\
L(\mathrm{D}_2)^1_2 &= \frac{i}{2}\left(d^1_2\overline{d^{\dot{2}}_{\dot{1}}} + d^2_2\overline{d^{\dot{1}}_{\dot{1}}} - d^1_1\overline{d^{\dot{2}}_{\dot{2}}} - d^2_1\overline{d^{\dot{1}}_{\dot{2}}}\right),\\
L(\mathrm{D}_2)^1_3 &= \frac{1}{2}\left(d^1_1\overline{d^{\dot{2}}_{\dot{1}}} + d^2_1\overline{d^{\dot{1}}_{\dot{1}}} - d^1_2\overline{d^{\dot{2}}_{\dot{2}}} - d^2_2\overline{d^{\dot{1}}_{\dot{2}}}\right),\\
L(\mathrm{D}_2)^2_0 &= \frac{i}{2}\left(d^1_1\overline{d^{\dot{2}}_{\dot{1}}} - d^2_1\overline{d^{\dot{1}}_{\dot{1}}} + d^1_2\overline{d^{\dot{2}}_{\dot{2}}} - d^2_2\overline{d^{\dot{1}}_{\dot{2}}}\right),\\
L(\mathrm{D}_2)^2_1 &= \frac{i}{2}\left(d^1_1\overline{d^{\dot{2}}_{\dot{2}}} - d^2_1\overline{d^{\dot{1}}_{\dot{2}}} + d^1_2\overline{d^{\dot{2}}_{\dot{1}}} - d^2_2\overline{d^{\dot{1}}_{\dot{1}}}\right),\\
L(\mathrm{D}_2)^2_2 &= \frac{1}{2}\left(d^1_1\overline{d^{\dot{2}}_{\dot{2}}} + d^2_2\overline{d^{\dot{1}}_{\dot{1}}} - d^1_2\overline{d^{\dot{2}}_{\dot{1}}} - d^2_1\overline{d^{\dot{1}}_{\dot{2}}}\right),\\
L(\mathrm{D}_2)^2_3 &= \frac{i}{2}\left(d^1_1\overline{d^{\dot{2}}_{\dot{1}}} - d^2_1\overline{d^{\dot{1}}_{\dot{1}}} - d^1_2\overline{d^{\dot{2}}_{\dot{2}}} + d^2_2\overline{d^{\dot{1}}_{\dot{2}}}\right),\\
L(\mathrm{D}_2)^3_0 &= \frac{1}{2}\left(d^1_1\overline{d^{\dot{1}}_{\dot{1}}} - d^2_1\overline{d^{\dot{2}}_{\dot{1}}} + d^1_2\overline{d^{\dot{1}}_{\dot{2}}} - d^2_2\overline{d^{\dot{2}}_{\dot{2}}}\right),\\
L(\mathrm{D}_2)^3_1 &= \frac{1}{2}\left(d^1_1\overline{d^{\dot{1}}_{\dot{2}}} - d^2_1\overline{d^{\dot{2}}_{\dot{2}}} + d^1_2\overline{d^{\dot{1}}_{\dot{1}}} - d^2_2\overline{d^{\dot{2}}_{\dot{1}}}\right),\\
L(\mathrm{D}_2)^3_2 &= \frac{i}{2}\left(d^1_2\overline{d^{\dot{1}}_{\dot{1}}} - d^2_2\overline{d^{\dot{2}}_{\dot{1}}} - d^1_1\overline{d^{\dot{1}}_{\dot{2}}} + d^2_1\overline{d^{\dot{2}}_{\dot{2}}}\right),\\
L(\mathrm{D}_2)^3_3 &= \frac{1}{2}\left(d^1_1\overline{d^{\dot{1}}_{\dot{1}}} - d^1_2\overline{d^{\dot{1}}_{\dot{2}}} - d^2_1\overline{d^{\dot{2}}_{\dot{1}}} + d^2_2\overline{d^{\dot{2}}_{\dot{2}}}\right).
\end{aligned}
\tag{36a}
$$

In addition, (28), (30), (33), and (34) imply

$$
\boldsymbol{X}^2 = G_{\alpha\beta}X^\alpha X^\beta = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2. \tag{37}
$$

Because of (29) and (37), Herm(2) is isomorphic to the Minkowski space, $\mathrm{FL}(4, \mathbb{R}) = \mathrm{O}^\uparrow_+(1, 3)$, and (22) coincides with the known 2-to-1 epimorphism $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{O}^\uparrow_+(1, 3)$ (Penrose and Rindler, 1986).

Let $\mathbb{FS}^2_\mathbb{R}$ be the realification of $\mathbb{FS}^2$ (see the book (Kostrikin and Manin, 1989) for the detailed information on the general realification procedure). Then, $\mathbb{FS}^2_\mathbb{R}$ is a four-dimensional vector space over $\mathbb{R}$ and its elements are Majorana 4-spinors. Indeed, setting

$$
\xi^1 = \xi^1_\mathbb{R} - i\xi^2_\mathbb{R}, \quad \xi^2 = \xi^3_\mathbb{R} - i\xi^4_\mathbb{R}, \quad \xi'^1 = \xi'^1_\mathbb{R} - i\xi'^2_\mathbb{R}, \quad \xi'^2 = \xi'^3_\mathbb{R} - i\xi'^4_\mathbb{R}, \tag{38}
$$

we obtain

$$\begin{aligned}\boldsymbol{\xi} &= \xi^a \boldsymbol{\epsilon}_a = \xi^1_{\mathbb{R}}\boldsymbol{\epsilon}_1 - \xi^2_{\mathbb{R}} i\boldsymbol{\epsilon}_1 + \xi^3_{\mathbb{R}}\boldsymbol{\epsilon}_2 - \xi^4_{\mathbb{R}} i\boldsymbol{\epsilon}_2,\\ \boldsymbol{\xi} &= \xi'^b \boldsymbol{\epsilon}'_b = \xi'^1_{\mathbb{R}}\boldsymbol{\epsilon}'_1 - \xi'^2_{\mathbb{R}} i\boldsymbol{\epsilon}'_1 + \xi'^3_{\mathbb{R}}\boldsymbol{\epsilon}'_2 - \xi'^4_{\mathbb{R}} i\boldsymbol{\epsilon}'_2\end{aligned} \tag{39}$$

for any $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2\}, \{\boldsymbol{\epsilon}'_1, \boldsymbol{\epsilon}'_2\} \in \mathbf{E}(\mathbb{FS}^2)$ and $\boldsymbol{\xi} \in \mathbb{FS}^2$; here $\xi^i_{\mathbb{R}}, \xi'^j_{\mathbb{R}} \in \mathbb{R}$ $(i, j = 1, 2, 3, 4)$. It follows from (39) that $\{\boldsymbol{\epsilon}_1, -i\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, -i\boldsymbol{\epsilon}_2\}$ and $\{\boldsymbol{\epsilon}'_1, -i\boldsymbol{\epsilon}'_1, \boldsymbol{\epsilon}'_2, -i\boldsymbol{\epsilon}'_2\}$ are bases in $\mathbb{FS}^2_{\mathbb{R}}$. Moreover, the substitution of (38) into (31) provides

$$\xi'^i_{\mathbb{R}} = M(\mathrm{D}_2)^i_j \xi^j_{\mathbb{R}}, \tag{40}$$

where $M(\mathrm{D}_2)^i_j \in \mathbb{R}$ and have the form

$$\begin{aligned}
M(\mathrm{D}_2)^1_1 &= \frac{1}{2}\left(\overline{d^1_1} + d^1_1\right), & M(\mathrm{D}_2)^3_1 &= \frac{1}{2}\left(\overline{d^2_1} + d^2_1\right),\\
M(\mathrm{D}_2)^1_2 &= \frac{i}{2}\left(\overline{d^1_1} - d^1_1\right), & M(\mathrm{D}_2)^3_2 &= \frac{i}{2}\left(\overline{d^2_1} - d^2_1\right),\\
M(\mathrm{D}_2)^1_3 &= \frac{1}{2}\left(\overline{d^1_2} + d^1_2\right), & M(\mathrm{D}_2)^3_3 &= \frac{1}{2}\left(\overline{d^2_2} + d^2_2\right),\\
M(\mathrm{D}_2)^1_4 &= \frac{i}{2}\left(\overline{d^1_2} - d^1_2\right), & M(\mathrm{D}_2)^3_4 &= \frac{i}{2}\left(\overline{d^2_2} - d^2_2\right),\\
M(\mathrm{D}_2)^2_1 &= \frac{i}{2}\left(d^1_1 - \overline{d^1_1}\right), & M(\mathrm{D}_2)^4_1 &= \frac{i}{2}\left(d^2_1 - \overline{d^2_1}\right),\\
M(\mathrm{D}_2)^2_2 &= \frac{1}{2}\left(d^1_1 + \overline{d^1_1}\right), & M(\mathrm{D}_2)^4_2 &= \frac{1}{2}\left(d^2_1 + \overline{d^2_1}\right),\\
M(\mathrm{D}_2)^2_3 &= \frac{i}{2}\left(d^1_2 - \overline{d^1_2}\right), & M(\mathrm{D}_2)^4_3 &= \frac{i}{2}\left(d^2_2 - \overline{d^2_2}\right),\\
M(\mathrm{D}_2)^2_4 &= \frac{1}{2}\left(d^1_2 + \overline{d^1_2}\right), & M(\mathrm{D}_2)^4_4 &= \frac{1}{2}\left(d^2_2 + \overline{d^2_2}\right).
\end{aligned} \tag{41}$$

It is evident that the matrix group $\mathrm{Maj}(4) = \{\|M(\mathrm{D}_2)^i_j\| \mid \mathrm{D}_2 \in \mathrm{SL}(2, \mathbb{C})\}$ is isomorphic to $\mathrm{SL}(2, \mathbb{C})$. Finally, using $\eta^1 = \eta^1_{\mathbb{R}} - i\eta^2_{\mathbb{R}}$, $\eta^2 = \eta^3_{\mathbb{R}} - i\eta^4_{\mathbb{R}}$, and (38), we can rewrite (32) as $[\boldsymbol{\xi}, \boldsymbol{\eta}] = \overline{\xi}\gamma^5\eta - i\overline{\xi}\eta$, where $\xi = (\xi^1_{\mathbb{R}}, \xi^2_{\mathbb{R}}, \xi^3_{\mathbb{R}}, \xi^4_{\mathbb{R}})^\top$ and $\eta = (\eta^1_{\mathbb{R}}, \eta^2_{\mathbb{R}}, \eta^3_{\mathbb{R}}, \eta^4_{\mathbb{R}})^\top$ are column matrices, the "⊤" mark denotes the matrix transposition, $\overline{\xi} = \xi^\top\gamma^0$ is a row matrix, and

$$\gamma^0 = \begin{pmatrix} 0 & 0 & i & 0\\ 0 & 0 & 0 & -i\\ -i & 0 & 0 & 0\\ 0 & i & 0 & 0\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & 0 & 0 & i\end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & i\\ 0 & 0 & i & 0\end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \qquad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{42}$$

are Dirac matrices in a Majorana representation (Majorana, 1937), which satisfy the standard conditions $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2g^{\alpha\beta}$ with $(g^{\alpha\beta}) = \mathrm{diag}\,(1, -1, -1, -1)$.

4. FINSLERIAN 3-SPINORS

In this section, we consider the nontrivial case of Finslerian N-spinors when $N = 3$. Besides, the algebraic structure of the group FL(9, $\mathbb{R}$) is also described here.

Let us begin with the following remark. For any $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3\}, \{\boldsymbol{\epsilon}'_1, \boldsymbol{\epsilon}'_2, \boldsymbol{\epsilon}'_3\} \in \mathbf{E}(\mathbb{FS}^3)$ and $\boldsymbol{\xi} = \xi^a\boldsymbol{\epsilon}_a = \xi'^b\boldsymbol{\epsilon}'_b \in \mathbb{FS}^3$, (4) implies $\xi'^a = d^a_b\xi^b$, where $\xi'^a, \xi^b \in \mathbb{C}$, $c^a_b d^b_c = \delta^a_c$, and $a, b, c = 1, 2, 3$. It is clear that C_3, $\mathrm{D}_3 \in \mathrm{SL}(3, \mathbb{C})$ and $\mathrm{D}_3 = \mathrm{C}_3^{-1}$ with the notation $\mathrm{C}_3 = \|c^a_b\|$, $\mathrm{D}_3 = \|d^a_b\|$. In the same way, (6) gives $[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}] = \varepsilon_{abc}\xi^a\eta^b\zeta^c$ for the scalar 3-product of arbitrary Finslerian 3-spinors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{\zeta}$ with respect to a basis $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3\} \in \mathbf{E}(\mathbb{FS}^3)$.

By analogy with the previous section, we set

$$\mathrm{E}^A = \frac{1}{2}\lambda^A, \qquad \mathrm{E}_B = \lambda_B, \tag{43}$$

where $A, B = 0, 1, \ldots, 8$, $\lambda^A = \lambda_A$ $(A \neq 8)$, $\lambda^8 = 2\lambda_8$, and

$$\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{44}$$

($\lambda_1, \lambda_2, \ldots, \lambda_7$ coincide with the corresponding Gell-Mann matrices). Since trace $(\lambda^A\lambda_B) = 2\delta^A_B$, the choice (43) guarantees correctness of (14). It follows from (13), (21), and (24) that

$$X'^A = L(\mathrm{D}_3)^A_B X^B \tag{45}$$

for any 9-vector $\boldsymbol{X} \in \mathrm{Herm}(3)$. Using (19), (43), and (44), we obtain

$$L(\mathrm{D}_3)^A_B = \frac{1}{2}\mathrm{trace}\big(\lambda^A\mathrm{D}_3\lambda_B\mathrm{D}_3^+\big). \tag{46}$$

In addition, (28), (30), (43), and (44) imply

$$\begin{aligned}\boldsymbol{X}^3 = G_{AB\Gamma}X^A X^B X^\Gamma &= [(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2]X^8\\ &- X^0[(X^4)^2 + (X^5)^2 + (X^6)^2 + (X^7)^2]\\ &+ 2X^1[X^4X^6 + X^5X^7] + 2X^2[X^5X^6 - X^4X^7]\\ &+ X^3[(X^4)^2 + (X^5)^2 - (X^6)^2 - (X^7)^2]. \end{aligned} \tag{47}$$

Because of (29), the Finslerian "scalar cube" (47) is forminvariant under the transformations (45) and (46) of the group FL(9, $\mathbb{R}$).

It is more or less clear that any matrix $\widehat{\mathrm{D}}_3 \in \mathrm{SL}(3, \mathbb{C})$ with $\hat{d}_3^3 \neq 0$ can be represented in a form of the product

$$\widehat{\mathrm{D}}_3 = \mathrm{D}_3^{(1)}\mathrm{D}_3^{(2)}\mathrm{D}_3^{(3)}\mathrm{D}_3^{(4)}, \tag{48}$$

where

$$\mathrm{D}_3^{(1)} = \begin{pmatrix} d_1^1 & d_2^1 & 0\\ d_1^2 & d_2^2 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad \mathrm{D}_3^{(2)} = \begin{pmatrix} 1 & 0 & d_3^1\\ 0 & 1 & d_3^2\\ 0 & 0 & 1\end{pmatrix},$$

$$\mathrm{D}_3^{(3)} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ d_1^3 & d_2^3 & 1\end{pmatrix}, \qquad \mathrm{D}_3^{(4)} = \begin{pmatrix} d & 0 & 0\\ 0 & d & 0\\ 0 & 0 & d^{-2}\end{pmatrix} \tag{49}$$

are SL(3, $\mathbb{C}$) matrices too. Because of (21), (48), and (49), we obtain the decomposition

$$L(\widehat{\mathrm{D}}_3) = L\left(\mathrm{D}_3^{(1)}\right)L\left(\mathrm{D}_3^{(2)}\right)L\left(\mathrm{D}_3^{(3)}\right)L\left(\mathrm{D}_3^{(4)}\right) \tag{50}$$

of the corresponding FL(9, $\mathbb{R}$) matrix $L(\widehat{\mathrm{D}}_3)$. Thus, (50) reduces a general FL(9, $\mathbb{R}$) transformation $X'^A = L(\widehat{\mathrm{D}}_3)^A_B X^B$ to a composition of four simpler ones induced by the matrices (49). These FL(9, $\mathbb{R}$) transformations will be explicitly described later.

Let $\xi^i_{\mathbb{R}} = X^{3+i}$ $(i = 1, 2, 3, 4)$. Then, with the help of (45) and (46), the FL(9, $\mathbb{R}$) transformation $X'^A = L(\mathrm{D}_3^{(1)})^A_B X^B$ is written in the following form:

$$\begin{aligned} X'^\alpha &= L(\mathrm{D}_2)^\alpha_\beta X^\beta,\\ \xi'^i_{\mathbb{R}} &= M(\mathrm{D}_2)^i_j \xi^j_{\mathbb{R}},\\ X'^8 &= X^8, \end{aligned} \tag{51}$$

where $\alpha, \beta = 0, 1, 2, 3$ and $i, j = 1, 2, 3, 4$. It is easy to see that the first line of (51) coincides with the Lorentz transformation (35), (36), (36a) of a 4-vector X^α, while the second line is the transformation (40), (41) of a Majorana 4-spinor $\xi^i_{\mathbb{R}}$.

Therefore, the transformations (51) form a 6-parametric non-Abelian subgroup of FL(9, $\mathbb{R}$).

Let $d_3^1 = \varepsilon^1 - i\varepsilon^2$, $d_3^2 = \varepsilon^3 - i\varepsilon^4$ be a parametrization of the complex matrix $\mathrm{D}_3^{(2)}$ from (49). Introducing the real column matrices $\varepsilon = (\varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^4)^\top$, $\xi = (X^4, X^5, X^6, X^7)^\top$ and using (42), (45), and (46), we can write the FL(9, $\mathbb{R}$) transformation $X'^A = L(\mathrm{D}_3^{(2)})^A_B X^B$ as

$$\begin{aligned} X'^\alpha &= X^\alpha + \overline{\varepsilon}\gamma^\alpha\xi + \frac{1}{2}\overline{\varepsilon}\gamma^\alpha\varepsilon X^8, \\ \xi' &= \xi + \varepsilon X^8, \\ X'^8 &= X^8, \end{aligned} \tag{52}$$

where $\alpha = 0, 1, 2, 3$ and $\overline{\varepsilon} = \varepsilon^\top\gamma^0$. Since $\varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^4 \in \mathbb{R}$, the transformations (52) form a 4-parametric Abelian subgroup of FL(9, $\mathbb{R}$).

Let $d_1^3 = \kappa^3 - i\kappa^4$, $d_2^3 = -\kappa^1 + i\kappa^2$ be a parametrization of the complex matrix $\mathrm{D}_3^{(3)}$ from (49). Introducing the real column matrices $\kappa = (\kappa^1, \kappa^2, \kappa^3, \kappa^4)^\top$, $\xi = (X^4, X^5, X^6, X^7)^\top$ and using (42), (45), and (46), we write the FL(9, $\mathbb{R}$) transformation $X'^A = L(\mathrm{D}_3^{(3)})^A_B X^B$ as

$$\begin{aligned} X'^\alpha &= X^\alpha, \\ \xi' &= -ig_{\alpha\beta}\gamma^\alpha\kappa X^\beta + \xi, \\ X'^8 &= g_{\alpha\beta}\overline{\kappa}\gamma^\alpha\kappa X^\beta + 2i\overline{\kappa}\xi + X^8, \end{aligned} \tag{53}$$

where $\alpha, \beta = 0, 1, 2, 3$, $\overline{\kappa} = \kappa^\top\gamma^0$, and $(g_{\alpha\beta}) = \mathrm{diag}(1, -1, -1, -1)$. Since $\kappa^1, \kappa^2, \kappa^3, \kappa^4 \in \mathbb{R}$, the transformations (53) form a 4-parametric Abelian subgroup of FL(9, $\mathbb{R}$).

Let $d = |d|e^{i\varphi} \neq 0$ be a parametrization of the complex matrix $\mathrm{D}_3^{(4)}$ from (49). Using (45) and (46), we represent the FL(9, $\mathbb{R}$) transformation $X'^A = L(\mathrm{D}_3^{(4)})^A_B X^B$ in the following form:

$$\begin{aligned} X'^\alpha &= |d|^2 X^\alpha, \\ \begin{pmatrix} X'^4 \\ X'^5 \end{pmatrix} &= |d|^{-1}\begin{pmatrix} \cos 3\varphi & \sin 3\varphi \\ -\sin 3\varphi & \cos 3\varphi \end{pmatrix}\begin{pmatrix} X^4 \\ X^5 \end{pmatrix}, \\ \begin{pmatrix} X'^6 \\ X'^7 \end{pmatrix} &= |d|^{-1}\begin{pmatrix} \cos 3\varphi & \sin 3\varphi \\ -\sin 3\varphi & \cos 3\varphi \end{pmatrix}\begin{pmatrix} X^6 \\ X^7 \end{pmatrix}, \\ X'^8 &= |d|^{-4} X^8, \end{aligned} \tag{54}$$

where $\alpha = 0, 1, 2, 3$. Since $|d| > 0$ and $\varphi \in \mathbb{R}$, the transformations (54) form a 2-parametric Abelian subgroup of FL(9, $\mathbb{R}$).

Thus, all of the four FL(9, $\mathbb{R}$) transformations corresponding to the matrices of the decomposition (50) have been explicitly described in (51), (52), (53), and (54). Finally, with the above notation, it is possible to rewrite (47) as $\boldsymbol{X}^3 = g_{\alpha\beta} X^\alpha X^\beta X^8 - g_{\alpha\beta} X^\alpha \bar{\xi} \gamma^\beta \xi$.

5. CONCLUSION

In the present paper, we have considered algebraic aspects of the Finslerian N-spinor theory. We formulated the general definitions of a Finslerian N-spinor and Finslerian N-spintensor of an arbitrary valency. It was shown that Finslerian N-spintensors of the valency $\left[\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}\right]$ were closely associated with the N^2-dimensional flat Finslerian space Herm(N). The metric on Herm(N) was characterized by the homogeneous algebraic form (30) of the Nth power. We also constructed the generalization (22) of the well-known epimorphism SL(2, $\mathbb{C}$) $\rightarrow$ $O_+^\uparrow(1, 3)$ and found that its kernel consisted of the N scalar matrices (23). In particular, it turned out that Finslerian 2-spinors coincided with standard Weyl spinors. In this connection, we recalled some essential information on Majorana 4-spinors as well. Finally, we considered properties of Finslerian 3-spinors and described the algebraic structure of the group FL(9, $\mathbb{R}$).

After this article had already been written, we learned about the works of Finkelstein (1986) and Finkelstein *et al.* (1986) in which *hyperspinors* and some of their properties were considered. David Finkelstein's hyperspinors actually coincide with Finslerian N-spinors for which we have developed the detailed algebraic theory here. We are grateful to Andrei Galiautdinov for attracting our attention to the works on hyperspinors.

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